Numerical methods for dynamic contact and dynamic fracture

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Introduction

We study two models:

- unilateral contact with or without Coulomb friction
- cohesive zone models with prescribed crack path

→ these models are characterized by set-valued boundary conditions

We adopt the following numerical approach:

- finite elements in space
- time-integration scheme

In this framework, we encounter some difficulties:

- dynamics + contact
- explicit schemes
Cohesive zone with prescribed crack path

1. The crack path is known in advance.
2. The bulk behavior is governed by linear elastodynamic equations.
3. The separation process at the interface fracture obeys a cohesive zone model.
Unilateral contact with or without Coulomb friction

Signorini problem: contact between a deformable solid and a rigid obstacle (small deformations).
Outline

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   - Introduction
   - Presentation of the modified mass method
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   - Coulomb friction case
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2. A semi-explicit scheme for cohesive zone models
   - Introduction
   - Modelling issues
   - A semi-explicit scheme
   - Numerical simulations (in progress)
   - Conclusion
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Introduction

Usual space-time discretizations in structural dynamics combine finite element space approximation and time-integration schemes.

Most methods exhibit spurious oscillations and/or poor energy conservation when applied to dynamic contact problems.

⇒ In [Khenous, Laborde and Renard, EJM A/Solids, 2008], the authors proposed a modification of the mass matrix which enables to overcome these difficulties.
Spurious oscillations

Example 1: impact of an elastic bar
A 1D elastic bar is dropped against a rigid ground without gravity force.

exact solution:

displacement at point A

contact pressure
Spurious oscillations

Example 1: impact of an elastic bar
A 1D elastic bar is dropped against a rigid ground without gravity force.

finite elements in space (standard mass) + Newmark scheme:

⇒ spurious oscillations
Poor energy conservation

Example 2: bounces of an elastic bar
A 1D elastic bar is dropped against a rigid ground with a gravity force.

exact solution:

```
\begin{align*}
\text{displacement at point A} & \\
\text{contact pressure} & \\
\end{align*}
```
Poor energy conservation

Example 2: bounces of an elastic bar
A 1D elastic bar is dropped against a rigid ground with a gravity force.

finite elements in space (standard mass) + Newmark scheme:

⇒ poor energy conservation
Why modify the mass matrix?

Impact law:

\[ \begin{align*}
\vec{v}_{-} - \vec{B} & = 0 \\
\vec{v}_{+} & = \vec{v}_{-} + \vec{B}
\end{align*} \]

⇒ an impact law is needed for the problem to be well-posed
Why modify the mass matrix?

contact condition

⇓

oscillations of the acceleration
at the contact boundary

⇓

oscillations of the contact pressure

⇓

perturbation of the whole structure
poor energy conservation
Why modify the mass matrix?

- Penalty contact condition, explicit schemes, dissipative schemes, stabilizations procedures
- Oscillations of the contact pressure
- Schemes where the contact pressure does not work
- No mass at the boundary
- Perturbation of the whole structure
- Poor energy conservation

Contact condition

- Oscillations of the acceleration at the contact boundary
- No mass at the boundary
- Schemes where the contact pressure does not work

Why modify the mass matrix?
Modified mass method

Form of the modified mass matrix

The entries corresponding to normal components at the contact boundary are set to zero.

\( N_d \) : number of dofs; \( N_c \) : number of nodes at the contact boundary; \( N_d^* = N_d - N_c \).

Standard mass: 
\[
\begin{pmatrix}
N_d^* & N_c \\
N_c & \begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}
\end{pmatrix}
\]

Modified mass: 
\[
\begin{pmatrix}
N_d^* & N_c \\
N_c & \begin{pmatrix}
M^{**} & 0 \\
0 & 0
\end{pmatrix}
\end{pmatrix}
\]

How to choose \( M^{**} \)?

- simplest choice: \( M^{**} = M_{11} \).
- choices preserving total mass, etc... [Khenous, Laborde and Renard, EJM A/Solids, 2008], [Hager, Hüeber and Wohlmuth, IJNME, 2008]

Time discretization

With the modified mass matrix, we obtain a space semi-discrete problem. This semi-discrete problem can then be discretized with various time-integration schemes.
Spurious oscillations

Example 1: impact of an elastic bar
A 1D elastic bar is dropped against a rigid ground without gravity force.

finite elements in space (standard mass) + Newmark scheme:

⇒ spurious oscillations

D. Doyen (CERMICS & EDF R&D)
Spurious oscillations

**Example 1 : impact of an elastic bar**
A 1D elastic bar is dropped against a rigid ground without gravity force.

finite elements in space (modified mass) + Newmark scheme :

⇒ no oscillations
Poor energy conservation

Example 2: bounces of an elastic bar
A 1D elastic bar is dropped against a rigid ground with a gravity force.

finite elements in space (standard mass) + Newmark scheme:

⇒ poor energy conservation
Poor energy conservation

Example 2: bounces of an elastic bar
A 1D elastic bar is dropped against a rigid ground with a gravity force.

finite elements in space (modified mass) + Newmark scheme:

⇒ good energy conservation
HHT scheme (dissipative scheme)

- Displacement at point A
- Contact pressure
- Displacement at point A
- Energy
Newmark scheme + stabilization step

![Graphs showing displacement at point A and contact pressure over time, energy over time.](image)
Modified mass method for contact problems

Presentation of the modified mass method

Newmark scheme + penalty contact condition

![Displacement at point A](image1)

![Contact pressure](image2)

![Displacement at point A](image3)

![Energy](image4)

D. Doyen (CERMICS & EDF R&D)
Midpoint scheme + contact condition in velocity

displacement at point A

contact pressure

D. Doyen (CERMICS & EDF R&D)
Space semi-discrete problem

We consider a finite element Signorini problem.

Let $V$ be the finite element space. The space $V$ can be decomposed as

$$V = V^* \oplus V^c,$$

where $V^c$ is spanned by normal displacements at the contact boundary and $V^*$ by the other dofs. A displacement $v$ can be decomposed as $v = v^* + v_c$.

We define the unilateral contact term $I_K : V^c \to \mathbb{R}$

$$I_K(v_c) = \begin{cases} 
0 & \text{if } v_c \in K \\
+\infty & \text{if } v_c \not\in K
\end{cases}$$

The function $I_K$ is non-differentiable but convex, and its domain is $K \cap V^c$. We can define its subgradient

$$\partial I_K(v_c).$$
Seek $u \in C^0([0, T]; K)$ such that $u_* \in C^2([0, T]; V^*)$, and

$$M^* \ddot{u}_* + A u + \lambda_c = L(t) \quad \text{a.e. in } [0, T],$$

$$\lambda_c \in \partial I_K(u_c),$$

with the initial conditions $u_*(0) = u_0^*$ and $\dot{u}_*(0) = v_0^*$ in $\Omega$.

$M^*$ : modified mass term; $A$ : stiffness term; $L(t)$ : external forces; $K$ : set of admissible displacements
Space semi-discrete problem

\[ M^* \ddot{u}_* + Au + \lambda_c = L(t) \quad \text{a.e. in } [0, T], \quad (1) \]
\[ \lambda_c \in \partial I_K(u_c), \quad (2) \]

Set \( L(t) = L^*(t) + L^c(t) \) and \( Au = A^* u + A^c u \).

Distinguishing components in \( V^* \) and \( V^c \), (1)-(2) can be turned into

\[ M^* \ddot{u}_* + A^* u = L^*(t), \quad \text{a.e. in } [0, T], \quad (3) \]
\[ A^c u_c + \partial I_K(u_c) \in -A^c u_* + L^c(t), \quad \text{a.e. in } [0, T]. \quad (4) \]

If we fix \( u_* \) and \( t \), there exists one and only one \( u_c \) satisfying (4).

Denote by \( q : [0, T] \times V^* \rightarrow V^c \cap K \) the map such that for a given \( u_* \) and \( t \), \( u_c = q(t, u_*) \) is the unique solution of (4). The map \( q \) is Lipschitz continuous with respect to its first and its second component.

Therefore (3)-(4) can be recast as

\[ M^* \ddot{u}_* + A^*(u_* + q(t, u_*)) = L^*(t), \quad \text{a.e. in } [0, T], \quad (5) \]
\[ u_c = q(t, u_*), \quad \forall t \in [0, T]. \quad (6) \]

\[ \Rightarrow \] The semi-discrete problem is equivalent to a Lipschitz system of ODEs.
Space semi-discrete problem

Existence and uniqueness [Khenous, Laborde and Renard, EJM A/Solids, 2008.]

The semi-discrete problem is equivalent to a Lipschitz system of ODEs. Therefore, the semi-discrete has a unique solution.

Furthermore, the variation of energy is equal to the work of the external forces.


For all \( t_0 \in [0, T] \), the following energy balance holds true:

\[
E(u(t_0)) - E(u(0)) = \int_{0}^{t_0} l(t, \dot{u}(t))dt,
\]

where \( E(v) = \frac{1}{2} \left( (M^* \dot{v}_*, \dot{v}_*) + (Av, v) \right) \).
Fully discrete problem

An implicit scheme

For simplicity, the interval $[0, T]$ is divided into $N$ equal subintervals of length $\Delta t$. We set $t^n = n\Delta t$ and denote by $u^n$, $v^n$, and $a^n$ the approximations of $u(t^n)$, $\dot{u}(t^n)$, and $\ddot{u}(t^n)$, respectively.

We discretize

- the elastodynamic part with an implicit Newmark scheme (trapezoidal rule),
- unilateral contact conditions in an implicit way.

Fully discrete problem

Seek $u^{n+1} \in V$, $v^{n+1}_{\ast} \in V_{\ast}$, and $a^{n+1}_{\ast} \in V_{\ast}$ such that

$$M_{\ast} a^{n+1}_{\ast} + A u^{n+1} + \lambda^{n+1} = L(t^{n+1}),$$  \hspace{1cm} (8)

$$\lambda^{n+1} \in \partial I_K(u^{n+1}_{c}),$$  \hspace{1cm} (9)

$$u^{n+1}_{\ast} = u^{n}_{\ast} + \Delta t \ v^{n}_{\ast} + \frac{\Delta t^2}{2} a^{n+\frac{1}{2}}_{\ast},$$  \hspace{1cm} (10)

$$v^{n+1}_{\ast} = v^{n}_{\ast} + \Delta t \ a^{n+\frac{1}{2}}_{\ast}.$$  \hspace{1cm} (11)
Fully discrete problem

A semi-explicit scheme

We discretize

- the elastodynamic part with a central difference scheme,
- unilateral contact conditions in an explicit way.

Seek $u^{n+1} \in V$ such that

$$M^* \frac{u^{n+1}_* - 2u^n_* + u^{n-1}_*}{\Delta t^2} + Au^n + \lambda^n_c = L(t^n),$$  \hspace{1cm} (12)

$$\lambda^n_c \in \partial l_K(u^n_c).$$  \hspace{1cm} (13)

At each time step,

- the displacements of the nodes in the interior of the domain are computed in an explicit way,
- the displacements of the nodes at the contact boundary are computed by solving a nonlinear problem (similar to a static contact problem).
Convergence to a continuous solution

The semi-discrete solutions converge, up to a subsequence, to a solution of the continuous problem in the case of a visco-elastic material.

Visco-elastic material (Kelvin-Voigt): \( \sigma = A : \epsilon + B : \dot{\epsilon} \)

Weak continuous formulation:

\[
\int_0^T -m(\dot{u}, \dot{v} - \dot{u}) + a(u, v - u) + b(\dot{u}, v - u) \, dt + \cdots \geq \int_0^T l(t, v - u) \, dt.
\]

1. Semi-discrete solutions \( (u_h) \)
2. A priori estimates
3. Compactness arguments, weak convergence \( u_h \rightharpoonup u \)
4. Verify that \( u \) is a solution of the continuous problem
Numerical simulations

- Bounces of a 2D disk against a rigid ground.
- Modified mass. Linear FE. Central differences.
- Primal-dual active set strategy.
- FreeFem++
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Energy. Comparison with another semi-explicit scheme.
Space semi-discrete problem

We consider a finite element Signorini problem.

Let $V$ be the finite element space. The space $V$ can be decomposed as

$$V = V^i \oplus V^f \oplus V^c$$

$$= V^* \oplus V^c$$

where $V^c$ is spanned by normal displacements at the contact boundary, $V^f$ by tangential displacements at the contact boundary, and $V^i$ by the interior dofs. A displacement $v$ can be decomposed as $v = v^* + v_c = v_i + v_f + v_c$.

We define the friction term $j : V \times V^f \rightarrow \mathbb{R}$ such that

$$j(v, w_f) = \int_{\Gamma^c} \mu |\sigma_n(v)| |w_f|.$$ 

The function $j$ is non-differentiable with respect to its second argument, but convex, and its domain is $V^f$. We can define its subgradient with respect to its second argument

$$\partial_2 j(v, w_f).$$
Space semi-discrete problem

Seek $u \in C^0([0, T]; K)$ such that $u_* \in C^1([0, T]; V^*)$, $\dot{u}_* \in AC([0, T]; V^*)$, and

$$M_* \ddot{u}_* + Au + \lambda_c + \lambda_f = L(t) \quad \text{a.e. in } [0, T],$$

$$\lambda_c \in \partial I_{K}(u_c),$$

$$\lambda_f \in \partial_2 j(u, \dot{u}_f),$$

with the initial conditions $u_*(0) = u^0_*$ and $\dot{u}_*(0) = v^0_*$ in $\Omega$.

$M^*$: modified mass term; $A$: stiffness term; $L(t)$: external forces; $K$: set of admissible displacements.
Space semi-discrete problem

\[ M^* \ddot{u}_* + Au + \lambda_c + \lambda_f = L(t) \quad \text{a.e. in } [0, T], \]  
\[ \lambda_c \in \partial I_K(u_c), \]  
\[ \lambda_f \in \partial_2 j(u, \dot{u}_f). \]  

Set \( L(t) = L^*(t) + L^c(t) \) and \( Au = A^* u + A^c u \).

Distinguishing components in \( V^* \) and \( V^c \), (15)-(17) can be turned into

\[ M^* \ddot{u}_* \in -A^* u - \partial_2 j(u, \dot{u}_f) + L^*(t) \quad \text{a.e. in } [0, T], \]  
\[ 0 \in -A^c u - \partial I_K(u_c) + L^c(t) \quad \text{a.e. in } [0, T], \]

and,

\[ M^* \ddot{u}_* \in -A^*(u_* + q(t, u_*)) - \partial_2 j(u_* + q(t, u_*), \dot{u}_f) + L^*(t), \quad \text{a.e. in } [0, T], \]  
\[ u_c = q(t, u_*), \quad \forall t \in [0, T]. \]

\( \Rightarrow \) The semi-discrete problem is equivalent to a differential inclusion (an upper semi-continuous differential inclusion with one-sided Lipschitz condition).
Space semi-discrete problem

Existence and uniqueness

The semi-discrete problem is equivalent to an upper semi-continuous differential inclusion with one-sided Lipschitz condition. Therefore, the semi-discrete has a unique solution.

Furthermore, the variation of energy is equal to the work of the external forces and friction forces.

Energy balance

For all $t_0 \in [0, T]$, the following energy balance holds true:

$$ E(u(t_0)) - E(u(0)) = \int_{0}^{t_0} \left\{ l(t, \dot{u}(t)) - j(u(t), \dot{u}_f(t)) \right\} dt, $$

(22)

where $E(\nu) = \frac{1}{2} \left( (M^* \dot{\nu}_*, \dot{\nu}_*) + (A\nu, \nu) \right)$. 
### Fully discrete problem

For simplicity, the interval $[0, T]$ is divided into $N$ equal subintervals of length $\Delta t$. We set $t^n = n\Delta t$ and denote by $u^n$, $v^n$, and $a^n$ the approximations of $u(t^n)$, $\dot{u}(t^n)$, and $\ddot{u}(t^n)$, respectively.

We discretize

- the elastodynamic part with an implicit Newmark scheme (trapezoidal rule),
- unilateral contact and friction conditions in an implicit way.

#### Fully discrete problem

Seek $u^{n+1} \in V$, $\nu^{n+1}_* \in V^*$, and $a^{n+1}_* \in V^*$ such that

\[ M^* a^{n+1}_* + A u^{n+1} + \lambda^{n+1}_f + \lambda^{n+1}_c = L(t^{n+1}), \quad (23) \]
\[ \lambda^{n+1}_c \in \partial I_K(u^{n+1}_c), \quad (24) \]
\[ \lambda^{n+1}_f \in \partial 2j(u^{n+1}, v^{n+1}_f), \quad (25) \]
\[ u^{n+1}_* = u^n_* + \Delta t \nu^n_* + \frac{\Delta t^2}{2} a^{n+1}_*/2, \quad (26) \]
\[ \nu^{n+1}_* = \nu^n_* + \Delta t a^{n+1}_*/2. \quad (27) \]
Fully discrete problem

The fully discrete problem is similar to a static Coulomb friction problem. It is well-known that this static problem may have several solutions. Here, owing to the inertial term, the problem is well-posed.

Existence and uniqueness for the fully discrete problem

The fully discrete problem has a unique solution under the CFL-condition

$$\frac{\Delta t}{h} \leq C,$$

where $h$ is the mesh size near the contact boundary.

For a fixed discretization in space, we prove also that the fully discrete solutions converge to the space semi-discrete solution when the time step tends to zero.

Convergence

The fully discrete solutions converge to the semi-discrete solution.
Numerical simulations

- Impact of a 2D body against a rigid ground. Tresca Friction.
- Modified mass. Linear FE. Newmark scheme.
- Primal-dual active set strategy.
- FreeFem++

(a) no friction (left); (b) large friction coefficient (right).
Numerical simulations

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(a) no friction (left); (b) large friction coefficient (right).
Conclusion

Advantages of the modified mass method:

- oscillation-free
- good energy conservation
- compatible with various time-integration schemes
- easy implementation

Future work:

- error analysis
- extension to large deformation problems
- extension to thin structures
Modified mass method for contact problems
- Introduction
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A semi-explicit scheme for cohesive zone models
- Introduction
- Modelling issues
- A semi-explicit scheme
- Numerical simulations (in progress)
- Conclusion
Introduction

The dynamic fracture is a fast phenomenon, of same order as the wave speeds.

It is natural to consider explicit time-integration schemes. However, combining an explicit time-integration scheme with set-valued boundary conditions is not straightforward.

⇒ We propose a semi-explicit time-integration scheme.
Crack speed

The dilatational, shear and Rayleigh wave speeds are

\[ c_d = \sqrt{\frac{E(1 - \nu)}{\rho(1 + \nu)(1 - 2\nu)}}, \quad c_s = \sqrt{\frac{E}{2\rho(1 + \nu)}}, \quad c_R \approx c_s \frac{0.862 + 1.14\nu}{1 + \nu}. \]

For instance, for a steel:

\[ c_d = 5801 \text{ m/s}, \quad c_s = 3101 \text{ m/s}, \quad c_R \approx 2872 \text{ m/s}. \]

**Theoretical result**: the limiting crack speed in mode I for Griffith’s model is the Rayleigh wave speed.

**Numerical results**: the Rayleigh wave speed can be found.

**Experimental results**: the crack speeds are far lower than Rayleigh wave speed.
Rate-dependent cohesive zone models

Two options:
- modify the constitutive relation in the bulk
- modify the cohesive law

⇒ rate-dependent cohesive zone models (depending on the opening rate)

Example: Zhou-Molinari-Shioya model. The toughness increases with the opening rate, the critical stress remaining constant. New parameter $\eta$.

\[
\begin{array}{ccc}
\eta = 0 & \eta = 0.005 & \eta = 0.01 \\
2773 \text{ m/s} & 2042 \text{ m/s} & 1471 \text{ m/s}
\end{array}
\]
Why choose an explicit approach?

- In linear elastodynamics the central difference scheme is stable under the CFL condition
  \[ c_d \frac{\Delta t}{\Delta x} \leq C. \]

- To capture accurately the fracture phenomenon, we need small time steps.

⇒ It is natural to consider an explicit time-integration scheme.
How to design explicit schemes for multi-valued BC?

Solutions proposed in the literature:

- regularized cohesive zone models

Specific treatments: initiation, contact, tangential cohesive force near 0.

⇒ We propose a semi-explicit scheme.
A semi-explicit scheme

Description of the scheme :
- a central difference scheme (explicit scheme) for the interior nodes
- enforcement of the cohesive conditions: implicit in the opening, explicit in the opening rate and maximal opening.

At each time step, we have to:
- compute the displacement of the interior nodes in an explicit way
- solve a nonlinear problem at the interface (similar to a static cohesive problem)
- update the opening rate and the maximal opening
Solving the non-linear problem at the interface

In the static case, cohesive problems may have several solutions (equivalent to a non-convex minimization).

In the dynamic case, owing to the inertial term, we can prove that the problem is well-posed under the condition

$$\frac{\Delta t^2}{h_c} \leq C,$$

where $h_c$ is the mesh size near the fracture interface.

\[ \frac{1}{\Delta t^2} Mv + Kv - L + R(v) \geq 0 \]
DCB
Plate with a hole
Conclusion

Advantages of the method:

- robust and general way of treating cohesive zone models
- moderate computational cost

Future work:

- coupling with implicit schemes